ON THE INSTABILITY OF EIGENVALUES

SYLVAIN GOLÉNIA

ABSTRACT. This is the proceeding of a talk given in Workshop on Differential Geometry and its applications at Alexandru Ioan Cuza University Iaşi, Romania, September 2–4, 2009. I explain how positive commutator estimates help in the analysis of embedded eigenvalues in a geometrical setting. Then, I will discuss the disappearance of eigenvalues in the perturbation theory and its relation with the Fermi golden rule.

1. INTRODUCTION

Let $\mathbb{H} := \{(x, y) \in \mathbb{R}^2, y > 0\}$ be the Poincaré half-plane and we endow it with the metric $g := y^{-2}(dx^2 + dy^2)$. Consider the group $\Gamma := PSL_2(\mathbb{Z})$. It acts faithfully on $\mathbb{H}$ by homographies, from the left. The interior of a fundamental domain of the quotient $\mathbb{H}\backslash\Gamma$ is given by $X := \{(x, y) \in \mathbb{H}, |x| < 1, x^2 + y^2 > 1\}$. Let $\mathscr{H} := L^2(X, g)$ be the set of L^2 integrable function acting on X, with respect to the volume element $dx\, dy/y^2$. Let $\mathcal{C}_b^\infty(X)$ be the restriction to X of the smooth bounded functions acting on $\mathbb{H}$ which are $\mathbb{C}$-valued and invariant under Γ. The (non-negative) Laplace operator is defined as the closure of

$$\Delta := -y^2(\partial_x^2 + \partial_y^2), \text{ on } \mathcal{C}_b^\infty(X).$$

It is a (unbounded) self-adjoint operator on $L^2(X)$. Using Eisenstein series, for instance, one sees that its essential spectrum is given by $[1/4, \infty)$ and that it has no singularly continuous spectrum, with respect to the Lebesgue measure. It is well-known that Δ has infinitely many eigenvalues accumulating at $+\infty$ and that every eigenspace is of finite dimension. We refer to [5] for an introduction to the subject.

We consider the Schrödinger operator $H_\lambda := \Delta + \lambda V$, where V is the multiplication by a bounded, real-valued function and $k \in \mathbb{R}$. We focus on an eigenvalue $k > 1/4$ of Δ and assume that the following hypothesis of *Fermi golden rule* holds true. Namely, there is $c_0 > 0$ so that:

$$\lim_{\varepsilon \to 0^+} PV\overline{P}\,\mathrm{Im}(H_0 - k + \mathrm{i}\varepsilon)^{-1}\overline{P}VP \geq c_0 P, \tag{1.1}$$

in the form sense and where $P := P_k$, the projection on the eigenspace of k, and $\overline{P} := 1 - P$. As P is of finite dimension, the limit can be taken in the weak or in the strong sense. At least formally, $\overline{P}\,\mathrm{Im}(H_0 - k + \mathrm{i}\varepsilon)^{-1}\overline{P}$ tends to the Dirac mass $\pi\delta_k(\overline{P}H_0)$. Therefore, the potential V couples the eigenspace of k and $\overline{P}H_0$ over k in a non-trivial way. This is a key assumption in the second-order perturbation theory of embedded eigenvalues, e.g., [13], and all the art is to prove that it implies there is $\lambda_0 > 0$ that H_λ has no eigenvalue in a neighborhood of k for $\lambda \in (0, |\lambda_0|)$.

In [4], one shows that generically the eigenvalues disappear under the perturbation of a potential (or of the metric) on a compact set. In this note, we are interested about the optimal decay at infinity of the perturbation given by a potential. Using the general result obtained in [3] and under a hypothesis of Fermi golden rule, one is only able to cover the assumption $VL^3 = o(1)$, as $y \to +\infty$, where L denotes the operator of multiplication by $L := (x, y) \mapsto 1 + \ln(y)$. We give the main result:

Theorem 1.1. *Let $k > 1/4$ be an L^2-eigenvalue of Δ. Suppose that $VL = o(1)$, as $y \to +\infty$ and that the Fermi golden rule* (1.1) *holds true, then there is $\lambda_0 > 0$, so that H_λ has no eigenvalue in a neighborhood of k, for all $\lambda \in (0, |\lambda_0|)$. Moreover, if $VL^{1+\varepsilon} = o(1)$, as $y \to +\infty$ for some $\varepsilon > 0$, then H_λ has no singularly continuous spectrum.*

We believe that the hypothesis $VL = o(1)$ is optimal in the scale of L. In our approach, we use the Mourre theory, see [1, 12] and establish a positive commutator estimate.

2. Idea of the proof

Standardly, for y large enough and up to some isometry $\mathcal{U}$, see for instance [6, 9, 10] the Laplace operator can be written as

$$\tilde{\Delta} = (-\partial_r^2 + 1/4) \otimes P_0 + \tilde{\Delta}(1 \otimes P_0^{\perp}) \tag{2.2}$$

on $\mathcal{C}_c^\infty\big((c, \infty), dr\big) \otimes \mathcal{C}^\infty(S^1)$, for some $c > 0$ and where P_0 is the projection on constant functions and $P_0^{\perp} := 1 - P_0$. The Friedrichs extension of the operator $\tilde{\Delta}(1 \otimes P_0^{\perp})$ has compact resolvent.

Then, as in [9, 10], we construct a conjugate operator. One chooses $\Phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\Phi(x) = x$ on $[-1, 1]$, and sets $\Phi_\Upsilon(x) := \Upsilon\Phi(x/\Upsilon)$, for $\Upsilon \geq 1$. Let $\tilde{\chi}$ be a smooth cut-off function being 1 for r big enough and 0 for r being close to c. We define on $\mathcal{C}_c^\infty\big((c, \infty) \times S^1\big)$ a micro-localized version of the generator of dilations:

$$S_{\Upsilon,0} := \tilde{\chi}\,\Big(\big(\Phi_\Upsilon(-\mathrm{i}\partial_r)r + r\Phi_\Upsilon(-\mathrm{i}\partial_r)\big) \otimes P_0\Big)\,\tilde{\chi}. \tag{2.3}$$

The operator $\Phi_\Upsilon(-\mathrm{i}\partial_r)$ is defined on the real line by $\mathscr{F}^{-1}\Phi_\Upsilon(\cdot)\mathscr{F}$, where $\mathscr{F}$ is the unitary Fourier transform. We also denote its closure by $S_{\Upsilon,0}$ and it is self-adjoint. In [6] for instance, one does not use a micro-localization and one is not able to deal with really singular perturbation of the metric as in [9, 10].

Now, one obtains

$$[\partial_r^2, \tilde{\chi}(\Phi_\Upsilon r + r\Phi_\Upsilon)\tilde{\chi}] = 4\tilde{\chi}\partial_r\Phi_\Upsilon\tilde{\chi} + \text{remainder}.$$

Using a cut-off function $\tilde{\mu}$ being 1 on the cusp and 0 for $y \leq 2$, we set

$$S_\Upsilon := \mathcal{U}^{-1}S_{\Upsilon,0}\,\mathcal{U}\,\widetilde{\mu} \tag{2.4}$$

This is self-adjoint in $L^2(X)$. Now by taking Υ big enough, one can show, as in [9, 10] that given an interval $\mathcal{J}$ around k, there exist $\varepsilon_\Upsilon > 0$ and a compact operator K_Υ such that the inequality

$$E_{\mathcal{J}}(\Delta)[\Delta, \mathrm{i}S_\Upsilon]E_{\mathcal{J}}(\Delta) \geq (4\inf(\mathcal{J}) - \varepsilon_\Upsilon)E_{\mathcal{J}}(\Delta) + E_{\mathcal{J}}(\Delta)K_\Upsilon E_{\mathcal{J}}(\Delta) \tag{2.5}$$

holds in the sense of forms, and such that ε_Υ tends to 0 as Υ goes to infinity. Here, $E_{\mathcal{J}}(\cdot)$ denotes the spectral measure above the interval $\mathcal{J}$.

Now, we apply $\overline{P}$ to the left and right of (2.5). Easily one has $\overline{P}E_{\mathcal{J}}(\Delta) = \overline{P}E_{\mathcal{J}}\big(\Delta\overline{P}\big)$. We get:

$$\begin{aligned}\overline{P}E_{\mathcal{J}}(\overline{P}\Delta)\big[\,\overline{P}\Delta, \mathrm{i}\overline{P}S_\Upsilon\overline{P}\,\big]E_{\mathcal{J}}(\overline{P}\Delta)\overline{P} \geq{}& (4\inf(\mathcal{J}) - \varepsilon_\Upsilon)\overline{P}E_{\mathcal{J}}(\Delta\overline{P})\overline{P}\\ &+ \overline{P}E_{\mathcal{J}}(\overline{P}\Delta)K_\Upsilon E_{\mathcal{J}}(\overline{P}\Delta)\overline{P}\end{aligned}$$

One can show that $\overline{P}S_\Upsilon\overline{P}$ is self-adjoint in $\overline{P}L^2(X)$ and that $\big[\,P\overline{\Delta}, \overline{P}S_\Upsilon\overline{P}\,\big]$ extends to a bounded operator.

We now shrink the size of the interval $\mathcal{J}$. As $\overline{P}\Delta$ has no eigenvalue in $\mathcal{J}$, then the operator $\overline{P}E_{\mathcal{J}}(\overline{P}\Delta)K_\Upsilon E_{\mathcal{J}}(\overline{P}\Delta)\overline{P}$ tends to 0 in norm. Therefore, by shrinking enough, one obtains a smaller interval $\mathcal{J}$ containing k and a constant $c > 0$ so that

$$\overline{P}E_{\mathcal{J}}(\overline{P}\Delta)\big[\,\overline{P}\Delta, \mathrm{i}\overline{P}S_\Upsilon\overline{P}\,\big]E_{\mathcal{J}}(\overline{P}\Delta)\overline{P} \geq c\overline{P}E_{\mathcal{J}}(\Delta\overline{P})\overline{P} \tag{2.6}$$

holds true in the form sense on $\overline{P}L^2(X)$. At least formally, the positivity on $\overline{P}L^2(X)$ of the commutator $\big[\,H_\lambda, \mathrm{i}\overline{P}S_\Upsilon\overline{P}\,\big]$, up to some spectral measure and to some small λ, should be a general fact and should not rely on the Fermi golden rule hypothesis.

We now try to extract some positivity on $PL^2(X)$. First, we set

$$R_\varepsilon := \big((H_0 - k)^2 + \varepsilon^2\big)^{-1/2},\ \overline{R_\varepsilon} := \overline{P}R_\varepsilon \text{ and } F_\varepsilon := \overline{R_\varepsilon}^2. \tag{2.7}$$

Note that $\varepsilon R_\varepsilon^2 = \mathrm{Im}(H_0 - k + \mathrm{i}\varepsilon)^{-1}$ and that R_ε commutes with P. Using (1.1), we get:

$$(c_1/\varepsilon)P \geq PV\overline{P}\,F_\varepsilon\,\overline{P}VP \geq (c_2/\varepsilon)P, \tag{2.8}$$

for $\varepsilon_0 > \varepsilon > 0$.

We follow an idea of [2], which was successfully used in [8, 11] and set

$$B_\varepsilon := \mathrm{Im}(\overline{R_\varepsilon}^2 VP).$$

It is a finite rank operator. Observe now that we gain some positivity as soon as $\lambda \neq 0$:

$$P[H_\lambda, \mathrm{i}\lambda B_\varepsilon]P = \lambda^2 PVF_\varepsilon VP \geq (c_2\lambda^2/\varepsilon)P. \tag{2.9}$$

It is therefore natural to modify the conjugate operator S_Υ to obtain some positivity on $PL^2(X)$. We set

$$\hat{S}_\Upsilon := \overline{P}S_\Upsilon\overline{P} + \lambda\theta B_\varepsilon. \tag{2.10}$$

It is self-adjoint on $\mathcal{D}(S_\Upsilon)$ and is diagonal with respect to the decomposition $\overline{P}L^2(X) \oplus PL^2(X)$.

Here $\theta > 0$ is a technical parameter. We choose ε and θ, depending on λ, so that $\lambda = o(\varepsilon)$, $\varepsilon = o(\theta)$ and $\theta = o(1)$ as λ tends to 0. We summarize this into:

$$|\lambda| \ll \varepsilon \ll \theta \ll 1, \text{ as } \lambda \text{ tends to } 0. \tag{2.11}$$

With respect to the decomposition $\overline{P}E_{\mathcal{J}}(\Delta)\oplus PE_{\mathcal{J}}(\Delta)$, as λ goes to 0, we have

$$E_{\mathcal{J}}(\Delta)\left[\lambda V, \mathrm{i}\overline{P}S_{\Upsilon}\overline{P}\,\right]E_{\mathcal{J}}(\Delta)=\begin{pmatrix} O(\lambda) & O(\lambda)\\ O(\lambda) & 0\end{pmatrix},$$

$$E_{\mathcal{J}}(\Delta)[\Delta, \mathrm{i}\lambda\theta B_{\varepsilon}]E_{\mathcal{J}}(\Delta)=\begin{pmatrix} 0 & O(\lambda\theta\varepsilon^{-1/2})\\ O(\lambda\theta\varepsilon^{-1/2}) & 0\end{pmatrix},$$

$$\text{and } E_{\mathcal{J}}(\Delta)[\lambda V, \mathrm{i}\lambda\theta B_{\varepsilon}]E_{\mathcal{J}}(\Delta)=\begin{pmatrix} O(\lambda^2\theta\varepsilon^{-3/2}) & O(\lambda^2\theta\varepsilon^{-3/2})\\ O(\lambda^2\theta\varepsilon^{-3/2}) & \lambda^2\theta F_{\varepsilon}\end{pmatrix}.$$

Now comes the delicate point. Under the condition (2.11) and by choosing $\mathcal{I}$, slightly smaller than $\mathcal{J}$, we use the previous estimates and a Schur Lemma to deduce:

$$E_{\mathcal{I}}(H_\lambda)[H_\lambda, \mathrm{i}\hat{S}_{\Upsilon}]E_{\mathcal{I}}(H_\lambda)\geq \frac{c\lambda^2\theta}{\varepsilon}E_{\mathcal{I}}(H_\lambda), \tag{2.12}$$

for some positive c and as λ tends to 0.

We mention that only the decay of VL is used to establish the last estimate. In fact, one uses that $[V, \mathrm{i}\hat{S}_{\Upsilon}](\Delta+1)^{-1}$ is a compact operator.

Now it is a standard use of the Mourre theory to deduce Theorem 1.1 and refer to [1], see [9, 10] for some similar application of the theory. For the absence of eigenvalue, one relies on the fact that given an eigenfunction f of H_λ w.r.t. an eigenvalue $\kappa\in\mathcal{I}$, one has:

$$\langle f, [H_\lambda, \mathrm{i}\hat{S}_{\Upsilon}]f\rangle = \langle f, [H_\lambda-\kappa, \mathrm{i}\hat{S}_{\Upsilon}]f\rangle = 0. \tag{2.13}$$

Then, one applies f on the right and on the left of (2.12) and infers that $f=0$ thanks to the fact that the constant $c\lambda^2\theta$ is non-zero.

In [9, 10], we prove that the C_0-group $(e^{\mathrm{i}S_{\Upsilon}t})_{t\in\mathbb{R}}$ stabilizes the domain $\mathcal{D}(H_\lambda)=\mathcal{D}(\Delta)$. By perturbation, we prove that this is also the case for $(e^{\mathrm{i}\hat{S}_{\Upsilon}t})_{t\in\mathbb{R}}$. Thanks to this property, we can expand the commutator of (2.13) in a legal way. This is known as the Virial theorem in the Mourre Theory, see [1, 12].

References

[1] W. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-Groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[2] V. Bach, J. Fröhlich, I.M. Sigal and A. Soffer: *Positive commutators and Spectrum of Pauli-Fierz Hamiltonian of Atoms and Molecules* , Commun. Math. Phys. **207** (3), 1999, 557–587.

[3] L. Cattaneo, G.M. Graf, and W. Hunziker *A general resonance theory based on Mourre's inequality* Ann. Henri Poincaré 7, No. 3, 583–601 (2006).

[4] Y. Colin de Verdière, *Pseudo-laplaciens II*, Ann. Inst. Fourier 33, No.2, 87–113 (1983).

[5] J. Fischer, *An approach to the Selberg trace formula via the Selberg zeta-function*, Lecture Notes in Mathematics, **1253**. Springer-Verlag, Berlin, 1987.

[6] R. Froese and P. D. Hislop, *Spectral analysis of second-order elliptic operators on noncompact manifolds*, Duke Math. J. **58** (1989), No. 1, 103-129.

[7] V. Georgescu and S. Golénia, *Decay Preserving Operators and stability of the essential spectrum,* J. Oper. Theory. **59** (2008), no. 1, 115–155.

[8] S. Golénia: *Positive commutators, Fermi golden rule and the spectrum of the zero temperature Pauli-Fierz hamiltonians*. J. of Funct. Anal., Vol. 256, Issue 8, April 2009, 2587-2620.

[9] S. Golénia and S. Moroianu, *Spectral analysis of magnetic Laplacians on conformally cusp manifolds*, Ann. H. Poincaré **9** (2008), 131–179.
[10] S. Golénia and S. Moroianu, *The spectrum of Schrödinger operators and Hodge Laplacians on conformally cusp manifolds*. To appear in Trans. A.M.S.
[11] M. Merlki, *Positive Commutators in Non-Equilibrium Statistical Mechanics*, Comm. Math. Phys. **223**, 327–362 (2001).
[12] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys. **91** (1981) 391–408.
[13] M. Reed and B. Simon, *Methods of modern mathematical physics I-IV: Fourier analysis, self- adjointness,* Academic Press, New York – San Francisco – London, 1975.

Institut de Mathématiques de Bordeaux, Université Bordeaux 1, 351, cours de la Libration, 33405 Talence cedex, France
E-mail address: sylvain.golenia@u-bordeaux1.fr